RELATIVISTIC COSMOLOGICAL HYDRODYNAMICS

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We investigate the relativistic cosmological hydrodynamic perturbations. We present the general large scale solutions of the perturbation variables valid for the general sign of three space curvature (K), the cosmological constant (Λ), and generally evolving background equation of state. The large scale evolution is characterized by *a conserved gauge invariant quantity* which is the same as a perturbed potential (or curvature) in the comoving gauge.

1 Relativistic Cosmological Hydrodynamics

1.1 Basic Equations

The evolution of a homogeneous and isotropic model universe is described by the following equations (the third equation follows from the first two):

$$H^2 = \frac{8\pi G}{3}\mu - \frac{K}{a^2} + \frac{\Lambda}{3}, \quad \dot{\mu} = -3H\left(\mu + p\right), \quad \dot{H} = -4\pi G(\mu + p) + \frac{K}{a^2}, \tag{1}$$

where $H \equiv \dot{a}/a$; μ and p are the energy density and the pressure, respectively. A set of equations describing small but general scalar type perturbations of the above world model is the following:

$$\dot{\delta}_v - 3H\mathrm{w}\delta_v = -\left(1+\mathrm{w}\right)\frac{k}{a}\frac{k^2-3K}{k^2}v_\chi - 2H\frac{k^2-3K}{a^2}\frac{\sigma}{\mu}, \tag{2}$$

$$\dot{v}_\chi + Hv_\chi = -\frac{k}{a}\varphi_\chi + \frac{k}{a\left(1+\mathrm{w}\right)}\left[c_s^2\delta_v + \frac{e}{\mu} - 8\pi G\left(1+\mathrm{w}\right)\sigma - \frac{2}{3}\frac{k^2-3K}{a^2}\frac{\sigma}{\mu}\right], \tag{3}$$

$$\frac{k^2-3K}{a^2}\varphi_\chi = 4\pi G\mu\delta_v, \tag{4}$$

$$\dot{\varphi}_\chi + H\varphi_\chi = -4\pi G\left(\mu+p\right)\frac{a}{k}v_\chi - 8\pi GH\sigma, \tag{5}$$

where $\mathrm{w} \equiv p/\mu$ and $c_s^2 \equiv \dot{p}/\dot{\mu}$. The fluid variables $\delta(\mathbf{k}, t)$, $v(\mathbf{k}, t)$, $e(\mathbf{k}, t)$ and $\sigma(\mathbf{k}, t)$ are the relative density perturbation ($\delta\mu/\mu$), the (frame independent) velocity, the entropic and anisotropic pressures, respectively. The metric variables $\varphi(\mathbf{k}, t)$ and

$\chi(\mathbf{k},t)$ are the perturbed part of the three space curvature and the shear, respectively. Eqs. (2-5) are written using the following gauge invariant combinations:

$$\delta_v \equiv \delta + 3(1+\mathrm{w})\frac{aH}{k}v, \quad \varphi_\chi \equiv \varphi - H\chi, \quad v_\chi \equiv v - \frac{k}{a}\chi. \tag{6}$$

δ_v is the same as δ in the comoving gauge ($v \equiv 0$), etc; we name the other gauge conditions as the uniform-curvature gauge ($\varphi \equiv 0$), the zero-shear gauge ($\chi \equiv 0$), the uniform-density gauge ($\delta \equiv 0$), etc. e and σ are gauge invariant.

1.2 *Closed Form Expressions*

Combining eqs. (2-5) we can derive a closed form expression for the δ_v as

$$\begin{aligned}
&\ddot{\delta}_v + \left(2 + 3c_s^2 - 6\mathrm{w}\right)H\dot{\delta}_v + \Bigg[c_s^2\frac{k^2}{a^2} - 4\pi G\mu\left(1 - 6c_s^2 + 6\mathrm{w} - 3\mathrm{w}^2\right) \\
&\qquad\qquad +12\left(\mathrm{w} - c_s^2\right)\frac{K}{a^2} + \left(3c_s^2 - 5\mathrm{w}\right)\Lambda\Bigg]\delta_v \\
&\quad = \frac{1+\mathrm{w}}{a^2H}\left[\frac{H^2}{a(\mu+p)}\left(\frac{a^3\mu}{H}\delta_v\right)^{\cdot}\right]^{\cdot} + c_s^2\frac{k^2}{a^2}\delta_v \\
&\quad = -\frac{k^2-3K}{a^2}\frac{1}{\mu}\left\{e + 2H\dot{\sigma} + 2\left[-\frac{1}{3}\frac{k^2}{a^2} + 2\dot{H} + 3\left(1+c_s^2\right)H^2\right]\sigma\right\}.
\end{aligned} \tag{7}$$

Notice that eq. (7) is valid for general K, Λ, and $p = p(\mu)$. The similar equation for φ_χ can be derived using eq. (4).

1.3 *General Solutions in the Large Scale*

On scales larger than the sound horizon (Jeans scale), ignoring the entropic and anisotropic pressures, eqs. (7,4) lead to a general integral form solution as

$$\begin{aligned}
\varphi_\chi(\mathbf{k},t) &= 4\pi G C(\mathbf{k})\frac{H}{a}\int_0^t \frac{a(\mu+p)}{H^2}dt + \frac{H}{a}d(\mathbf{k}) \\
&= C(\mathbf{k})\left[1 - \frac{H}{a}\int_0^t a\left(1 - \frac{K}{\dot{a}^2}\right)dt\right] + \frac{H}{a}d(\mathbf{k}),
\end{aligned} \tag{8}$$

where $C(\mathbf{k})$ and $d(\mathbf{k})$ are integration constants corresponding to the growing and decaying modes, respectively. δ_v and v_χ follow from eqs. (4,5), respectively, as:

$$\delta_v(\mathbf{k},t) = \frac{k^2-3K}{4\pi G\mu a^2}\varphi_\chi(\mathbf{k},t), \tag{9}$$

$$v_\chi(\mathbf{k},t) = -\frac{k}{4\pi G(\mu+p)a^2}\left\{C(\mathbf{k})\left[\frac{K}{\dot{a}} - \dot{H}\int_0^t a\left(1 - \frac{K}{\dot{a}^2}\right)dt\right] + \dot{H}d(\mathbf{k})\right\}. \tag{10}$$

For $K = 0 = \Lambda$ and $\mathrm{w} =$ constant, thus $a \propto t^{2/[3(1+\mathrm{w})]}$, eqs. (8-10) become:

$$\begin{aligned}
\varphi_\chi(\mathbf{k}, t) &= \frac{3(1+\mathrm{w})}{5+3\mathrm{w}} C(\mathbf{k}) + \frac{2}{3(1+\mathrm{w})} \frac{1}{at} d(\mathbf{k}) \\
&\propto \text{constant}, \quad t^{-\frac{5+3\mathrm{w}}{3(1+\mathrm{w})}} \quad \propto \quad \text{constant}, \quad a^{-\frac{5+3\mathrm{w}}{2}}, \\
\delta_v(\mathbf{k}, t) &\propto t^{\frac{2(1+3\mathrm{w})}{3(1+\mathrm{w})}}, \quad t^{-\frac{1-\mathrm{w}}{1+\mathrm{w}}} \quad \propto \quad a^{1+3\mathrm{w}}, \quad a^{-\frac{3}{2}(1-\mathrm{w})}, \\
v_\chi(\mathbf{k}, t) &\propto t^{\frac{1+3\mathrm{w}}{3(1+\mathrm{w})}}, \quad t^{-\frac{4}{3(1+\mathrm{w})}} \quad \propto \quad a^{\frac{1+3\mathrm{w}}{2}}, \quad a^{-2}. \qquad (11)
\end{aligned}$$

1.4 A Conserved Quantity

φ_v (the curvature fluctuation in the comoving gauge) is known to be conserved in the large scale limit independently of the changes in the background equation of state. Since we have $\varphi_v = \varphi_\chi - (aH/k)v_\chi$ from eq. (6), eqs. (8,10) lead to

$$\begin{aligned}
\varphi_v(\mathbf{k}, t) &= C(\mathbf{k}) \left\{ 1 + \frac{K}{a^2} \frac{1}{4\pi G(\mu+p)} \left[1 - \frac{H}{a} \int_0^t a \left(1 - \frac{K}{\dot{a}^2} \right) dt \right] \right\} \\
&\quad + \frac{K}{a^2} \frac{H/a}{4\pi G(\mu+p)} d(\mathbf{k}). \qquad (12)
\end{aligned}$$

For $K = 0$ (but for a general Λ) we have

$$\varphi_v(\mathbf{k}, t) = C(\mathbf{k}), \qquad (13)$$

with the *vanishing* decaying mode. Thus, for $K = 0$, φ_v is conserved for the generally time varying equation of state, $p = p(\mu)$. This conservation property of the curvature variable in a certain gauge also applies to the models based on a minimally coupled scalar field or even on classes of generalized gravity theories [1].

2 Newtonian Correspondences

After a thorough investigation of the behavior of variables in the pressureless limit in various gauge conditions we have identifed the following correspondences with the Newtonian perturbation variables which are valid in a general scale:

$$\delta_v \leftrightarrow \delta, \quad \frac{k^2 - 3K}{k^2} v_\chi \leftrightarrow \delta v, \quad -\frac{k^2 - 3K}{k^2} \varphi_\chi \leftrightarrow \delta\Phi, \qquad (14)$$

where δ, δv, and $\delta\Phi$ in the right-hand-sides are the relative density fluctuation ($\delta \equiv \delta\varrho/\varrho$), the velocity fluctuation, and the potential fluctuation, in the Newtonian context, respectively. Eqs. (2-4) can be compared with the continuity, the momentum conservation, and the Poisson's equations in the Newtonian context, respectively. Parts of these correspondences were studied in [2]. A complete version of this work is presented in [3].

Acknowledgments

This work was supported by the KOSEF, Grant No. 95-0702-04-01-3 and through the SRC program of SNU-CTP.

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